Suggested solutions of assignment 2 :

(1) By Thm 2 in Handout-1 and an exercise in assignment 1, we see that α must be part of a circular helix. Since

$$\begin{cases} \kappa = \dfrac{a}{a^2+b^2} \\ \tau = \dfrac{b}{a^2+b^2} \end{cases} \Rightarrow \tau^2 + \kappa^2 = \frac{1}{a^2+b^2} \Rightarrow \begin{cases} a = \dfrac{\kappa}{\kappa^2+\tau^2} \\ b = \dfrac{\tau}{\kappa^2+\tau^2} \end{cases}$$

where we assume the circular helix $\alpha(t) = (a\cos t, a\sin t, bt)$ for some $a>0$, $t \in \mathbb{R}$. (A circular helix is determined by a, b.) Therefore, κ_0, τ_0 determine a circular helix. #

(2) $(\Leftarrow)$: If $\exists$ constant unit vector u s.t $\langle T, u\rangle = \text{Constant}$, then we may write

$$\langle T, u\rangle = \cos\theta, \text{ where } \theta \text{ is some constant.}$$

Differentiation $\Rightarrow$

$$0 = \langle T', u\rangle = \langle \kappa N, u\rangle$$

$$\Rightarrow \quad \langle N, u\rangle = 0$$

$$\Rightarrow \quad u = \langle u, T\rangle T + \langle u, B\rangle B$$

$$= \cos\theta\, T + \sin\theta\, B$$

Differentiation $\Rightarrow$

$$0 = \cos\theta\, T' + \sin\theta\, B'$$

$$= \cos\theta\, \kappa N + \sin\theta\, (-\tau N)$$

$= (\kappa \cos\theta - \tau \sin\theta) N$

$$\Rightarrow \quad \frac{\tau}{\kappa} = \cot\theta = \text{Constant}.$$

($\Leftarrow$) If τ/κ = Constant, then $\exists$ constant θ s.t

$$\tau/\kappa = \cot\theta$$

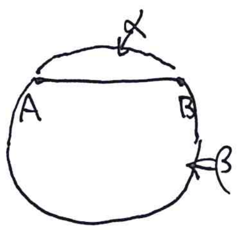
Def $u = \cos\theta\, T + \sin\theta\, B$

Then $|u| = 1$

$u' = 0$

$\langle T, u \rangle = \cos\theta$ = Constant. $\#$

(3) Let C be the circle s.t AB is a chord of it and AB divide C into α and β where α has length l. Consider any curve γ with length l joining A, B s.t γ, AB form a Jordan curve. Then we consider the closed curve $[\gamma, \beta]$, let γ vary and β be fixed. By a general isoperimetric inequality (See Rmk 2 in Do Carmo's Differential Geometry of Curves and Surfaces Page 35), we see that when $\gamma = \alpha$ $[\gamma, \beta]$ encloses the maximal area, this actually shows



what we require. (You should give a ~~more~~ detailed discussion.)

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W.LOG, $\alpha(t)$ is unit speed and the disc has centre at origin.

(4) Let $f(t) = \|\alpha(t)\|^2$, we consider the farthest point (w.r.t the origin) on $\alpha(t)$, say $\alpha(t_0)$. So we have $\begin{cases} f'(t_0) = 0 \\ f''(t_0) \leq 0 \end{cases}$. (We can parametrize s.t. $t_0 \in (0, l)$, where l = length of α.)

Note that $f'' = 2(\langle \alpha'', \alpha \rangle + \langle \alpha', \alpha' \rangle)$, so $\langle \alpha''(t_0), \alpha(t_0) \rangle \leq -1$.

So $K(t_0) \cdot d \cos\theta \leq -1$ where $K(t_0)$ is the curvature of α at $t = t_0$,

$d = \text{dist}(0, \alpha(t_0))$, θ is the angle between $\alpha''(t_0)$ and $\alpha(t_0)$.

So $\cos\theta < 0$. So $K(t_0) \geq \frac{-1}{d\cos\theta} \geq \frac{-1}{r\cos\theta} \geq \frac{1}{r}$.

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